Sheaf Algorithms Using the Exterior Algebra

Wolfram Decker and David Eisenbud

In this chapter we explain constructive methods for computing the cohomology of a sheaf on a projective variety. We also give a construction for the Beilinson monad, a tool for studying the sheaf from partial knowledge of its cohomology. Finally, we give some examples illustrating the use of the Beilinson monad.

1 Introduction

In this chapter V denotes a vector space of finite dimension $n+1$ over a field K with dual space $W = V^*$, and $S = \mathrm{Sym}_K(W)$ is the symmetric algebra of W, isomorphic to the polynomial ring on a basis for W. We write E for the exterior algebra on V. We grade S and E by taking elements of W to have degree 1, and elements of V to have degree -1. We denote the projective space of 1-quotients of W (or of lines in V) by $\mathbf{P}^n = \mathbf{P}(W)$.

Serre's sheafification functor $M \mapsto \tilde{M}$ allows one to consider a coherent sheaf on $\mathbf{P}(W)$ as an equivalence class of finitely generated graded S-modules, where we identify two such modules M and M' if, for some r, the truncated modules $M_{\geq r}$ and $M'_{\geq r}$ are isomorphic. A free resolution of M, sheafified, becomes a resolution of $\tilde{M}$ by sheaves that are direct sums of line bundles on $\mathbf{P}(W)$ – that is, a description of $\tilde{M}$ in terms of homogeneous matrices over S. Being able to compute syzygies over S one can compute the cohomology of $\tilde{M}$ starting from the minimal free resolution of M (see [16], [40] and Remark 3.2 below).

The Bernstein-Gel'fand-Gel'fand correspondence (BGG) is an isomorphism between the derived category of bounded complexes of finitely generated S-modules and the derived category of bounded complexes of finitely generated E-modules or of certain "Tate resolutions" of E-modules. In this chapter we show how to effectively compute the Tate resolution $\mathbf{T}(\mathcal{F})$ associated to a sheaf $\mathcal{F}$, and we use this construction to give relatively cheap computations of the cohomology of $\mathcal{F}$.

It turns out that by applying a simple functor to the Tate resolution $\mathbf{T}(\mathcal{F})$ one gets a finite complex of sheaves whose homology is the sheaf $\mathcal{F}$ itself. This complex is called a *Beilinson monad* for $\mathcal{F}$. The Beilinson monad provides a powerful method for getting information about a sheaf from partial knowledge of its cohomology. It is a representation of the sheaf in terms of direct sums of (suitably twisted) bundles of differentials and homomorphisms between these bundles, which are given by homogeneous matrices over E.

The following recipe for computing the cohomology of a sheaf is typical of our methods: Suppose that $\mathcal{F} = \tilde{M}$ is the coherent sheaf on $\mathbf{P}(W)$ associated to a finitely generated graded S-module $M = \oplus M_i$. To compute the cohomology of $\mathcal{F}$ we consider a sequence of free E-modules and maps

$$\mathbf{F}(M): \quad \cdots \longrightarrow F^{i-1} \xrightarrow{\phi_{i-1}} F^i \xrightarrow{\phi_i} F^{i+1} \longrightarrow \cdots .$$

Here we set $F^i = M_i \otimes_K E$ and define $\phi_i : F^i \longrightarrow F^{i+1}$ to be the map taking $m \otimes 1 \in M_i \otimes_K E$ to

$$\sum_j x_j m \otimes e_j \in M_{i+1} \otimes V \subset F^{i+1},$$

where $\{x_j\}$ and $\{e_j\}$ are dual bases of W and V respectively. It turns out that $\mathbf{F}(M)$ is a complex; that is, $\phi_i \phi_{i-1} = 0$ for every i (the reader may easily check this by direct computation; a proof without indices is given in [18]). If we regard M_i as a vector space concentrated in degree i, so that F^i is a direct sum of copies of $E(-i)$, then these maps are homogeneous of degree 0.

We shall see that if s is a sufficiently large integer then the truncation of the Tate resolution

$$F^s \xrightarrow{\phi_s} F^{s+1} \longrightarrow \cdots$$

is exact and is thus the minimal injective resolution of the finitely generated graded E-module $P_s = \ker \phi_{s+1}$. (In fact any value of s greater than the Castelnuovo-Mumford regularity of M will do.)

Because the number of monomials in E in any given degree is small compared to the number of monomials of that degree in the symmetric algebra, it is relatively cheap to compute a free resolution of P_s over E, and thus to compute the graded vector spaces $\mathrm{Tor}_t^E(P_s, K)$. Our algorithm exploits the fact, proved in [18], that the j^{th} cohomology $\mathrm{H}^j\mathcal{F}$ of $\mathcal{F}$ in the Zariski topology is isomorphic to the degree $-n-1$ part of $\mathrm{Tor}_{s-j}^E(P_s, K)$; that is,

$$\mathrm{H}^j\mathcal{F} \cong \mathrm{Tor}_{s-j}^E(P_s, K)_{-n-1}.$$

In addition, the linear parts of the matrices in the complex $\mathbf{T}(\mathcal{F})$ determine the graded S-modules

$$\mathrm{H}^j_*\mathcal{F} := \oplus_{i\in\mathbb{Z}} \mathrm{H}^j\mathcal{F}(i) \ .$$

In many cases this is the fastest known method for computing cohomology.

Section 2 of this paper is devoted to a sketch of the Eisenbud-Fløystad-Schreyer approach to the Bernstein-Gel'fand-Gel'fand correspondence, and the computation of cohomology, together with *Macaulay 2* programs that carry it out, is explained in Section 3.

The remainder of this paper is devoted to an explanation of the Beilinson monad, how to compute it in *Macaulay 2*, and what it is good for. This technique has played an important role in the construction and study of vector bundles and varieties. In the typical application one constructs or classifies monads in order to construct or classify sheaves.

The BGG correspondence and Beilinson's monad were originally formulated in the language of derived categories, and the proofs were rather complicated. The ideas of Eisenbud-Fløystad-Schreyer exposed above allow, for the first time, an explanation of these matters on a level that can be understood by an advanced undergraduate.

The Beilinson monad is similar in spirit to the technique of free resolutions. That theory essentially describes arbitrary sheaves by comparing them with direct sums of line bundles. In the Beilinson technique, one uses a different set of "elementary" sheaves, direct sums of exterior powers of the tautological sub-bundle. Beilinson's remarkable observation was that this comparison has a much more direct connection with cohomology than does the free resolution method.

Sections 4 and 5 are introductory in nature. In Section 4 we begin with a preparatory discussion of the necessary vector bundles on projective space and their cohomology. In Section 5 we define monads, a generalization of resolutions. We give a completely elementary account which constructs the Beilinson monad in a very special case, following ideas of Horrocks, and we use this to sketch part of one of the first striking applications of monads: the classification of stable rank 2 vector bundles on the projective plane by Barth, Hulek and Le Potier.

In Section 6 we give the construction of Eisenbud-Fløystad-Schreyer for the Beilinson monad in general. This is quite suitable for computation, and we give *Macaulay 2* code that does this job.

A natural question for the student at this point is: "Why should I bother learning Beilinson's theorem, what is it good for?" In section 7, we describe two more explicit applications of the theory developed. In the first, the classification of elliptic conic bundles in $\mathbf{P}^4$, computer algebra played a significant role, demonstrating that several published papers contained serious mistakes by constructing an example they had excluded! Using the routines developed earlier in the chapter we give a simpler account of the crucial computation.

In the second application, the construction of abelian surfaces in $\mathbf{P}^4$ and the related Horrocks-Mumford bundles, computer algebra allows one to greatly shorten some of the original arguments made. As the reader will see, everything follows easily with computation, once a certain 2×5 matrix of exterior monomials, given by Horrocks and Mumford, has been written down. One might compare the computations here with the original paper of Horrocks and Mumford [25] (for the cohomology) and the papers by Manolache [32] and Decker [13] (for the syzygies) of the Horrocks-Mumford bundle. A great deal of effort, using representation theory, was necessary to derive results that can be computed in seconds using the *Macaulay 2* programs here. Much more theoretical effort, however, is needed to derive classification results.

Another application of the construction of the Beilinson complex (in a slightly more general setting) is to compute Chow forms of varieties; see [19].

Perhaps the situation is similar to that in the beginning of the 1980's when it became clear that syzygies could be computed by a machine. Though syzygies had been used theoretically for many years it took quite a while until the practical computation of syzygies lead to applications, too, mostly through the greatly increased ability to study examples.

A good open problem of this sort is to extend and make more precise the very useful criterion given in 4.4: Can the reader find a necessary and sufficient condition to replace the necessary condition for surjectivity given there? How about a criterion for exactness?

2 Basics of the Bernstein-Gel'fand-Gel'fand Correspondence

In this section we describe the basic idea of the BGG correspondence, introduced in [8]. For a more complete treatment along the lines given here, see the first section of [18].

As a simple example of the construction given in Section 1, consider the case $M = S = \mathrm{Sym}_K(W)$. The associated complex, made from the homogeneous components $\mathrm{Sym}_i(W)$ of S, has the form

$$\mathbf{F}(S): \quad E \longrightarrow W \otimes E \longrightarrow \mathrm{Sym}_2(W) \otimes E \longrightarrow \cdots,$$

where we regard $\mathrm{Sym}_i W$ as concentrated in degree i. It is easy to see that the kernel of the first map, $E \longrightarrow W \otimes E$, is exactly the socle $\bigwedge^{n+1} V \subset E$, which is a 1-dimensional vector space concentrated in degree $-n-1$. In fact $\mathbf{F}(S)$ is the minimal injective resolution of this vector space. If we tensor with the dual vector space $\bigwedge^{n+1} W$ (which is concentrated in degree $n+1$), we obtain the minimal injective resolution of the vector space $\bigwedge^{n+1} W \otimes \bigwedge^{n+1} V$, which may be identified canonically with the residue field K of E. This resolution is called the *Cartan resolution* of K. To write it conveniently, we set $\omega_E = \bigwedge^{n+1} W \otimes E$. The socle of ω_E is K. Since E is injective (as well as projective) as an E-module, the same goes for ω_E, so ω_E is the injective envelope of the residue class field K and we have $\omega_E = \mathrm{Hom}_K(E, K)$. Thus we can write the injective resolution of the residue field as

$$\mathbf{R}(S): \quad \omega_E \longrightarrow W \otimes \omega_E \longrightarrow \mathrm{Sym}_2(W) \otimes \omega_E \longrightarrow \cdots,$$

or again as

$$\mathrm{Hom}_K(E, K) \longrightarrow \mathrm{Hom}_K(E, W) \longrightarrow \mathrm{Hom}_K(E, \mathrm{Sym}_2(W)) \longrightarrow \cdots.$$

Taking our cue from this situation, our primary object of study in the case of an arbitrary finitely generated graded S-module $M = \oplus M_i$ will be the complex

$$\mathbf{R}(M): \quad \cdots \longrightarrow M_i \otimes \omega_E \longrightarrow M_{i+1} \otimes \omega_E \longrightarrow \cdots,$$

which will have a more natural grading than $\mathbf{F}(M)$; in any case, it differs from $\mathbf{F}(M)$ only by tensoring over K with the one-dimensional K-vector space $\bigwedge^{n+1} W$, concentrated in degree $n+1$, and thus has the same basic properties. (Writing $\mathbf{R}(M)$ in terms of Hom as above suggests that the functor $\mathbf{R}$ might have a left adjoint, and indeed there is a left adjoint that produces linear free complexes over S from graded E-modules. $\mathbf{R}$ and its left adjoint are used to construct the isomorphisms of derived categories in the BGG correspondence; see [18] for a treatment in this spirit.)

An important fact for us is that the complex $\mathbf{R}(M)$ is eventually exact (and thus

$$F^i \xrightarrow{\phi_i} F^{i+1} \longrightarrow \cdots$$

is the minimal injective resolution of $\ker \phi_i$ when $i \gg 0$). It turns out that the point at which exactness sets in is a well-known invariant, the Castelnuovo-Mumford regularity of M, whose definition we briefly recall:

If $M = \oplus M_i$ is a finitely generated graded S-module then for all large integers r the submodule $M_{\geq r} \subset M$ is generated in degree r and has a *linear free resolution*; that is, its first syzygies are generated in degree $r+1$, its second syzygies in degree $r+2$, etc. (see [17, chapter 20]). The *Castelnuovo-Mumford regularity* of M is the least integer r for which this occurs.

Theorem 2.1 ([18]). *Let M be a finitely generated graded S-module of Castelnuovo-Mumford regularity r. The complex $\mathbf{R}(M)$ is exact at $\operatorname{Hom}_K(E, M_i)$ for all $i \geq s$ if and only if $s > r$.* ☐

More generally, it is shown in [18] that the components of the cohomology of the complex $\mathbf{R}(M)$ can be identified with the Koszul cohomology of M. An equivalent result was stated in [10].

For instance, it is not hard to show that if M is of finite length, then the regularity of M is the largest i such that $M_i \neq 0$. Let us verify Theorem 2.1 directly in a simple example:

Example 2.2. Let $S = K[x_0, x_1, x_2]$, and let $M = S/(x_0^2, x_1^2, x_2^2)$. The module $M_{\geq 3} = K \cdot x_0x_1x_2$ is a trivial S-module, and its resolution is the Koszul complex on x_0, x_1 and x_2, which is linear. Thus the Castelnuovo-Mumford regularity of M is ≤ 3. On the other hand $M_{\geq 2}$ is, up to twist, isomorphic to the dual of $S/(x_0, x_1, x_2)^2$, and it follows that the resolution of $M_{\geq 2}$ has the form

$$0 \longrightarrow S(-6) \longrightarrow 6S(-4) \longrightarrow 8S(-3) \longrightarrow 3S(-2),$$

which is not linear, so the Castelnuovo-Mumford regularity of M is exactly 3. Note that the regularity is larger than the degrees of the generators and relations of M—in general it can be much larger.

Over E the linear free complex corresponding to M has the form

$$\cdots \to 0 \to M_0 \otimes \omega_E \to M_1 \otimes \omega_E \to M_2 \otimes \omega_E \to M_3 \otimes \omega_E \to 0 \to \cdots ,$$

where all the terms not shown are 0. Using the isomorphism $\omega_E \cong E(-3)$ this can be written (non-canonically) as

$$0 \longrightarrow E(-3) \xrightarrow{\begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix}} 3E(-2) \xrightarrow{\begin{pmatrix} 0 & e_2 & e_1 \\ e_2 & 0 & e_0 \\ e_1 & e_0 & 0 \end{pmatrix}} 3E(-1) \xrightarrow{\begin{pmatrix} e_0 & e_1 & e_2 \end{pmatrix}} E \longrightarrow 0.$$

One checks easily that this complex is inexact at every non-zero term (despite its resemblance to a Koszul complex), verifying Theorem 2.1. ☐

Another case in which everything can be checked directly occurs when M is the homogeneous coordinate ring of a point:

Example 2.3. Take $M = S/I$ where I is generated by a codimension 1 space of linear forms in W, so that I is the homogeneous ideal of a point $p \in \mathbf{P}(W)$. The free resolution of M is the Koszul complex on n linear forms, so M is 0-regular. As M_i is 1-dimensional for every i the terms of the complex $\mathbf{R}(M)$ are all rank 1 free E-modules. One easily checks that $\mathbf{R}(M)$ takes the form

$$\mathbf{R}(M): \quad \omega_E \xrightarrow{a} \omega_E(-1) \xrightarrow{a} \omega_E(-2) \xrightarrow{a} \cdots,$$

where $a \in V = W^*$ is a linear functional that vanishes on all the linear forms in I; that is, a is a generator of the one-dimensional subspace of V corresponding to the point p. As for any linear form in E, the annihilator of a is generated by a, and it follows directly that the complex $\mathbf{R}(M)$ is acyclic in this case. ☐

We present two *Macaulay 2* functions, `symExt` and `bgg`, which compute a differential of the complex $\mathbf{R}(M)$ for a finitely generated graded module M defined over some polynomial ring $S = K[x_0, \ldots, x_n]$ with variables x_i of degree 1. Both functions expect as an additional input the name of an exterior algebra E with the same number $n+1$ of generators, also supposed to be of degree 1 (and NOT -1). This convention, which makes the cohomology diagrams more naturally looking when printed in *Macaulay 2*, necessitates the adjustment of degrees in the second half of the programs.

The first of the functions, `symExt`, takes as input a matrix m with linear entries, which we think of as a presentation matrix for a positively graded S-module $M = \oplus_{i \geq 0} M_i$, and returns a matrix representing the map $M_0 \otimes \omega_E \to M_1 \otimes \omega_E$ which is the first differential of the complex $\mathbf{R}(M)$.

```
i1 : symExt = (m,E) ->(
          ev := map(E,ring m,vars E);
          mt := transpose jacobian m;
          jn := gens kernel mt;
          q  := vars(ring m)**id_(target m);
          ans:= transpose ev(q*jn);
          --now correct the degrees:
          map(E^{(rank target ans):1}, E^{(rank source ans):0},
              ans));
```

If M is a module whose presentation is not linear in the sense above, we can still apply `symExt` to a high truncation of M:

```
i2 : S=ZZ/32003[x_0..x_2];

i3 : E=ZZ/32003[e_0..e_2,SkewCommutative=>true];

i4 : M=coker matrix{{x_0^2, x_1^2}};

i5 : m=presentation truncate(regularity M,M);

               4       8
o5 : Matrix S  <--- S

i6 : symExt(m,E)

o6 = {-1} | e_2 e_1 e_0 0   |
     {-1} | 0   e_2 0   e_0 |
     {-1} | 0   0   e_2 e_1 |
     {-1} | 0   0   0   e_2 |

               4       4
o6 : Matrix E  <--- E
```

The function `symExt` is a quick-and-dirty tool which requires little computation. If it is called on two successive truncations of a module the maps it produces may NOT compose to zero because the choice of bases is not consistent. The second function, `bgg`, makes the computation in such a way that the bases are consistent, but does more computation to achieve this end. It takes as input an integer i and a finitely generated graded S-module M, and returns the i^{th} map in $\mathbf{R}(M)$, which is an "adjoint" of the multiplication map between M_i and M_{i+1}.

```
i7 : bgg = (i,M,E) ->(
          S :=ring(M);
          numvarsE := rank source vars E;
          ev:=map(E,S,vars E);
          f0:=basis(i,M);
          f1:=basis(i+1,M);
          g :=((vars S)**f0)//f1;
          b:=(ev g)*((transpose vars E)**(ev source f0));
          --correct the degrees (which are otherwise
          --wrong in the transpose)
          map(E^{(rank target b):i+1},E^{(rank source b):i}, b));
```

For instance, in Example 2.2:

```
i8 : M=cokernel matrix{{x_0^2, x_1^2, x_2^2}};

i9 : bgg(1,M,E)

o9 = {-2} | e_1 e_0 0   |
     {-2} | e_2 0   e_0 |
     {-2} | 0   e_2 e_1 |

               3       3
o9 : Matrix E  <--- E
```

3 The Cohomology and the Tate Resolution of a Sheaf

Given a finitely generated graded S-module M we construct a (doubly infinite) E-free complex $\mathbf{T}(M)$ with vanishing homology, called the *Tate resolution* of M, as follows: Let r be the Castelnuovo-Mumford regularity of M. The truncation $\mathbf{T}^{>r}(M)$, the part of $\mathbf{T}(M)$ with cohomological degree $> r$, is $\mathbf{R}(M_{>r})$. We complete this to an exact complex by adjoining a minimal projective resolution of the kernel of $\mathrm{Hom}_K(E, M_{r+1}) \to \mathrm{Hom}_K(E, M_{r+2})$.

If, for example, M has finite length as in Example 2.2, the Tate resolution of M is the complex

$$\cdots \to 0 \to 0 \to 0 \to \cdots .$$

At the opposite extreme, take $M = S$, the free module of rank 1. Since S has regularity 0, it follows that $\mathbf{R}(S)$ is an injective resolution of the residue field K of E. Applying the exact functor $\mathrm{Hom}_K(\text{—}, K)$, and using the fact that it carries $\omega_E = \mathrm{Hom}_K(E, K)$ back to E, we see that the Tate resolution $\mathbf{T}(S)$ is the first row of the diagram

$$\begin{tikzcd} \cdots \arrow[r] & W^* \otimes E \arrow[r] & E \arrow[rr] \arrow[dr] & & \omega_E \arrow[r] & W \otimes \omega_E \arrow[r] & \cdots \\ & & & K \arrow[ur] & & & \end{tikzcd}$$

Another simple example occurs in the case where M is the homogeneous coordinate ring of a point $p \in \mathbf{P}(W)$. The complex $\mathbf{R}(M)$ constructed in Example 2.3 is periodic, so it may be simply continued to the left, giving

$$\mathbf{T}(M): \quad \cdots \xrightarrow{a} \omega_E(i) \xrightarrow{a} \omega_E(i-1) \xrightarrow{a} \cdots ,$$

where again $a \in V = W^*$ is a non-zero linear functional vanishing on the linear forms in the ideal of p.

For arbitrary M, by the results of the previous section, $\mathbf{R}(M_{>r})$ has no homology in cohomological degree $> r+1$, so $\mathbf{T}(M)$ could be constructed by a similar recipe from any truncation $\mathbf{R}(M_{>s})$ with $s \geq r$. Thus the Tate resolution depends only on the sheaf $\tilde{M}$ on $\mathbf{P}(W)$ corresponding to M. We sometimes write $\mathbf{T}(M)$ as $\mathbf{T}(\tilde{M})$ to emphasize this point.

Using the *Macaulay 2* function `symExt` of the last section, one can compute any finite piece of the Tate resolution.

```
i10 : tateResolution = (m,E,loDeg,hiDeg)->(
          M := coker m;
          reg := regularity M;
          bnd := max(reg+1,hiDeg-1);
          mt  := presentation truncate(bnd,M);
          o   := symExt(mt,E);
          --adjust degrees, since symExt forgets them
          ofixed   :=  map(E^{(rank target o):bnd+1},
                      E^{(rank source o):bnd},
                      o);
          res(coker ofixed, LengthLimit=>max(1,bnd-loDeg+1)));
```

`tateResolution` takes as input a presentation matrix m of a finitely generated graded module M defined over some polynomial ring $S = K[x_0, \ldots, x_n]$ with variables x_i of degree 1, the name of an exterior algebra E with the same number $n+1$ of generators, also supposed to be of degree 1, and two integers, say l and h. If r is the regularity of M, then `tateResolution(m,E,l,h)` computes the piece

$$\mathbf{T}^{\,l}(M) \to \cdots \to \mathbf{T}^{\,\max(r+2,h)}(M)$$

of $\mathbf{T}(M)$. For instance, for the homogeneous coordinate ring of a point in the projective plane:

```
i11 : m = matrix{{x_0,x_1}};

              1       2
o11 : Matrix S  <--- S

i12 : regularity coker m

o12 = 0

i13 : T = tateResolution(m,E,-2,4)

       1      1      1      1      1      1      1
o13 = E  <-- E  <-- E  <-- E  <-- E  <-- E  <-- E

      0      1      2      3      4      5      6

o13 : ChainComplex

i14 : betti T

o14 = total: 1 1 1 1 1 1 1
         -4: 1 1 1 1 1 1 1

i15 : T.dd_1

o15 = {-4} | e_2 |

              1       1
o15 : Matrix E  <--- E
```

For arbitrary M we have $M_i = \mathrm{H}^0\tilde{M}(i)$ for large i, so the corresponding term of the complex $\mathbf{T}(\tilde{M})$ with cohomological degree i is $M_i \otimes \omega_E = \mathrm{H}^0(\tilde{M}(i)) \otimes \omega_E$. The following result generalizes this to a description of all the terms of the Tate resolution, and gives the formula for the cohomology described in the introduction.

Theorem 3.1 ([18]). *Let M be a finitely generated graded S-module. The term of the complex $\mathbf{T}(M) = \mathbf{T}(\tilde{M})$ with cohomological degree i is*

$$\oplus_j \mathrm{H}^j\tilde{M}(i-j) \otimes \omega_E \ ,$$

where $\mathrm{H}^j\tilde{M}(i-j)$ is regarded as a vector space concentrated in degree $i-j$, so that the summand $\mathrm{H}^j\tilde{M}(i-j)\otimes\omega_E$ is isomorphic to a direct sum of copies *of* $\omega_E(j-i)$. *Moreover the subquotient complex*

$$\cdots \to \mathrm{H}^j\tilde{M}(i-j)\otimes\omega_E \to \mathrm{H}^j\tilde{M}(i+1-j)\otimes\omega_E \to \cdots$$

is $\mathbf{R}(\mathrm{H}^j_*(\tilde{M}(-j)))(j)$ *(up to twists and shifts it is* $\mathbf{R}(\mathrm{H}^j_*\tilde{M})$*.)* $\square$

Thus each cohomology group of each twist of the sheaf $\tilde{M}$ occurs (exactly once) in a term of $\mathbf{T}(M)$. When we compute a part of $\mathbf{T}(M)$, we are computing the sheaf cohomology of various twists of the associated sheaf together with maps which describe the S-module structure of $\mathrm{H}^j_*\tilde{M}$ in the sense that the linear maps in this complex are adjoints of the multiplication maps that determine the module structure (the multiplication maps themselves could be computed by a function similar to `bgg`). The higher degree maps in the complex $\mathbf{T}(M)$ determine certain higher cohomology operations, which we understand only in very special cases (see [19]).

If $M = \operatorname{coker} m$, then `betti tateResolution(m,E,l,h)` prints the dimensions $\mathrm{h}^j\tilde{M}(i-j) = \dim \mathrm{H}^j\tilde{M}(i-j)$ for $\max(r+2,h) \geq i \geq l$, where r is the regularity of M. Truncating the Tate resolution if necessary allows one to restrict the size of the output.

```
i16 : sheafCohomology = (m,E,loDeg,hiDeg)->(
           T := tateResolution(m,E,loDeg,hiDeg);
           k := length T;
           d := k-hiDeg+loDeg;
           if d > 0 then
              chainComplex apply(d+1 .. k, i->T.dd_(i))
           else T);
```

The expression `betti sheafCohomology(m,E,l,h)` prints a cohomology table for $\tilde{M}$ of the form

$$\begin{matrix} \mathrm{h}^0\tilde{M}(h) & \ldots & \mathrm{h}^0\tilde{M}(l) \\ \mathrm{h}^1\tilde{M}(h-1) & \ldots & \mathrm{h}^1\tilde{M}(l-1) \\ \vdots & & \vdots \\ \mathrm{h}^n\tilde{M}(h-n) & \ldots & \mathrm{h}^n\tilde{M}(l-n)\ . \end{matrix}$$

As a simple example we consider the cotangent bundle on projective 3-space (see the next section for the Koszul resolution of this bundle):

```
i17 : S=ZZ/32003[x_0..x_3];

i18 : E=ZZ/32003[e_0..e_3,SkewCommutative=>true];
```

The cotangent bundle is the cokernel of the third differential of the Koszul complex on the variables of S.

```
i19 : m=koszul(3,vars S);

                6       4
o19 : Matrix S  <--- S
```

```
i20 : regularity coker m

o20 = 2

i21 : betti tateResolution(m,E,-6,2)

o21 = total: 45 20 6 1 4 15 36 70 120 189 280
         -4: 45 20 6 . .  .  .  .   .   .   .
         -3:  .  . . 1 .  .  .  .   .   .   .
         -2:  .  . . . .  .  .  .   .   .   .
         -1:  .  . . . 4 15 36 70 120 189 280

i22 : betti sheafCohomology(m,E,-6,2)

o22 = total: 6 1 4 15 36 70 120 189 280
         -2: 6 . .  .  .  .   .   .   .
         -1: . 1 .  .  .  .   .   .   .
          0: . . .  .  .  .   .   .   .
          1: . . 4 15 36 70 120 189 280
```

Of course these two results differ only in the precise point of truncation.

Remark 3.2. There is also a built-in sheaf cohomology function `HH` in *Macaulay 2* which is based on the algorithms in [16]. These algorithms are often much slower than `sheafCohomology`. To access it, first execute

```
M=sheaf coker m;
```

and pick integers j and d. Then

```
HH^j(M(>=d))
```

returns the truncated j^{th} cohomology module $\mathrm{H}^j_{i\geq d}\tilde{M}$. In the above example of the cotangent bundle $\mathcal{F}$ on projective 3-space we obtain the Koszul presentation of $H^1\mathcal{F} \cong K$ considered as an S-module sitting in degree 0:

```
i23 : M=sheaf coker m;

i24 : HH^1(M(>=0))

o24 = cokernel | x_3 x_2 x_1 x_0 |

                                1
o24 : S-module, quotient of S
```

□

The Tate resolutions of sheaves are, as the reader may easily check, precisely the doubly infinite, graded, exact complexes of finitely-generated free E-modules which are "eventually linear" on the right, in an obvious sense. What about other doubly exact graded free complexes? For example what if we take the dual of the Tate resolution of a sheaf? In general it will not be eventually linear. What is it?

To explain this we must generalize the construction of $\mathbf{R}(M)$: If

$$M^\bullet : \quad \cdots \longrightarrow M^{i+1} \longrightarrow M^i \longrightarrow M^{i-1} \longrightarrow \cdots$$

is a complex of S-modules, then applying the functor $\mathbf{R}$ gives a complex of free complexes over E. By changing some signs we get a double complex. In general the associated total complex is not minimal; but at least if $M^\bullet$ is a bounded complex then, just as one produces the unique minimal free resolution of a module from any free resolution, we can construct a unique minimal complex from it. We call this minimal complex $\mathbf{R}(M^\bullet)$. (See [18] for more information. This construction is a necessary part of interpreting the BGG correspondence as an equivalence of derived categories.)

Again if $M^\bullet$ is a bounded complex of finitely generated modules, then as before one shows that $\mathbf{R}(M^\bullet)$ is exact from a certain point on, and so we can form the Tate resolution $\mathbf{T}(M^\bullet)$ by adjoining a free resolution of a kernel. Once again, the Tate resolution depends only on the bounded complex of coherent sheaves $\mathcal{F}^\bullet$ associated to $M^\bullet$, and we write $\mathbf{T}(\mathcal{F}^\bullet) = \mathbf{T}(M^\bullet)$.

A variant of the theorem of Bernstein, Gel'fand and Gel'fand shows that every minimal graded doubly infinite exact sequence of finitely generated free E-modules is of the form $\mathbf{T}(\mathcal{F}^\bullet)$ for some complex of coherent sheaves $\mathcal{F}^\bullet$, unique up to quasi-isomorphism. The terms of the Tate resolution can be expressed using hypercohomology by a formula like that of Theorem 3.1.

One way that interesting complexes of sheaves arise is through duality. For simplicity, write $\mathcal{O}$ for the structure sheaf $\mathcal{O}_{\mathbf{P}(W)}$. If $\mathcal{F} = \tilde{M}$ is a sheaf on $\mathbf{P}(W)$ then the derived functor $RHom(\mathcal{F}, \mathcal{O})$ may be computed by applying the functor $Hom(—, \mathcal{O})$ to a sheafified free resolution of M; it's value is thus a complex of sheaves rather than an individual sheaf.

We can now identify the dual of the Tate resolution:

Theorem 3.3. $\mathrm{Hom}_K(\mathbf{T}(\mathcal{F}), K) \cong \mathbf{T}(RH\mathrm{om}(\mathcal{F}, \mathcal{O}))[1]$. □

Here the [1] denotes a shift by one in cohomological degree. For example, take $\mathcal{F} = \mathcal{O}$. We have $RH\mathrm{om}(\mathcal{O}, \mathcal{O}) = \mathcal{O}$. The Tate resolution is given by

$$\mathbf{T}(\mathcal{O}): \quad \begin{array}{ccccccc} \cdots & \longrightarrow & E & \longrightarrow & \omega_E & \longrightarrow & \cdots \\ & & -1 & & 0 & & \end{array}$$

where the number under each term is its cohomological degree. Taking into account $\omega_E = \mathrm{Hom}_K(E, K)$, the dual of the Tate resolution is thus

$$\mathrm{Hom}_K(\mathbf{T}(\mathcal{O}), K): \quad \begin{array}{ccccccc} \cdots & \longleftarrow & \omega_E & \longleftarrow & E & \longleftarrow & \cdots \\ & & 1 & & 0 & & \end{array}$$

which is the same as $\mathbf{T}(\mathcal{O})[1]$. A completely analogous computation gives the proof of Theorem 3.3 if $\mathcal{F} = \mathcal{O}(a)$ for some a, and the general case follows by taking free resolutions.

4 Cohomology and Vector Bundles

In this section we first recall how vector bundles, direct sums of line bundles, and bundles of differentials can be characterized among all coherent sheaves on $\mathbf{P}(W)$ in terms of cohomology (as usual we do not distinguish between vector bundles and locally free sheaves). Then we describe the homomorphisms between the suitably twisted bundles of differentials in terms of the exterior algebra E. This description plays an important role in the context of Beilinson monads.

Vector bundles on $\mathbf{P}(W)$ are characterized by a criterion of Serre [39] which can be formulated as follows: A coherent sheaf $\mathcal{F}$ on $\mathbf{P}(W)$ is locally free if and only if its module of sections $\mathrm{H}^0_*(\mathcal{F})$ is finitely generated and its *intermediate cohomology modules* $\mathrm{H}^j_*\mathcal{F}$, $1 \le j \le n-1$, are of finite length.

From a cohomological point of view, the simplest vector bundles are the direct sums of line bundles. Every vector bundle on the projective line splits into a direct sum of line bundles by Grothendieck's splitting theorem (see [37]). Induction yields Horrocks' splitting theorem (see [5]): A vector bundle on $\mathbf{P}(W)$ splits into a direct sum of line bundles if and only if its intermediate cohomology vanishes (originally, this theorem was proved as a corollary to a more general result, see [23] and [42]).

Just a little bit more complicated are the bundles of differentials. To fix our notation in this context we write $\mathcal{O} = \mathcal{O}_{\mathbf{P}(W)}$, $W \otimes \mathcal{O}$ for the trivial bundle on $\mathbf{P}(W)$ with fiber W, $U = \Omega_{\mathbf{P}(W)}(1)$ for the cotangent bundle twisted by 1, and

$$U^i = \bigwedge^i U = \bigwedge^i(\Omega_{\mathbf{P}(W)}(1)) = \Omega^i_{\mathbf{P}(W)}(i)$$

for the i^{th} bundle of differentials twisted by i; in particular $U^0 = \mathcal{O}$, $U^n \cong \mathcal{O}(-1)$, and $U^i = 0$ if $i < 0$ or $i > n$.

Remark 4.1. For each $0 \le i \le n$ the pairing

$$U^i \otimes U^{n-i} \xrightarrow{\wedge} U^n \cong \mathcal{O}(-1)$$

induces an isomorphism

$$U^{n-i} \cong (U^i)^*(-1)\,. \qquad \square$$

The fiber of U at the point of $\mathbf{P}(W)$ corresponding to the line $\langle a \rangle \subset V$ is the subspace $(V/\langle a \rangle)^* \subset W$. Thus U fits into the short exact sequence

$$0 \to U \to W \otimes \mathcal{O} \to \mathcal{O}(1) \to 0\ .$$

In fact, U is the *tautological subbundle* of $W \otimes \mathcal{O}$. Taking exterior powers, we get the short exact sequences

$$0 \to U^{i+1} \to \bigwedge^{i+1} W \otimes \mathcal{O} \to U^i \otimes \mathcal{O}(1) \to 0\ .$$

Twisting the i^{th} sequence by $-i-1$, and gluing them together we get the exact sequence

$$0 \longrightarrow \bigwedge^{n+1} W \otimes \mathcal{O}(-n-1) \longrightarrow \cdots \longrightarrow \bigwedge^0 W \otimes \mathcal{O} \longrightarrow 0\ .$$

This sequence is the sheafification of the Koszul complex, which is the free resolution of the "trivial" graded S-module $K = S/(W)$.

Remark 4.2. By taking cohomology in the short exact sequences above we find that

$$\mathrm{H}^j_* U^i = \begin{cases} K(i) & j = i, \\ 0 & j \neq i, \end{cases} \qquad 1 \leq i, j \leq n-1 \ ,$$

where $K(i) = (S/(W))(i)$. Conversely, every vector bundle $\mathcal{F}$ on $\mathbf{P}(W)$ with this intermediate cohomology is *stably equivalent* to U^i; that is, there exists a direct sum $\mathcal{L}$ of line bundles such that $\mathcal{F} \cong U^i \oplus \mathcal{L}$. This follows by comparing the sheafified Koszul complex with the minimal free resolution of the dual bundle $\mathcal{F}^*$. $\square$

In what follows we describe the homomorphisms between the various U^i, $0 \leq i \leq n$. Note that since $U = U^1 \subset W \otimes \mathcal{O}$ each element of $V = \mathrm{Hom}_K(W, K)$ induces a homomorphism $U^1 \to U^0$ which is the composite

$$U^1 \subset W \otimes \mathcal{O} \to K \otimes \mathcal{O} = \mathcal{O} = U^0.$$

Similarly, using the diagonal map of the exterior algebra $U^i = \bigwedge^i U \to U \otimes U^{i-1}$, each element of V induces a homomorphism $U^i \to U^{i-1}$ which is the composite

$$U^i \to U \otimes U^{i-1} \to W \otimes U^{i-1} \to K \otimes U^{i-1} = U^{i-1}.$$

It is not hard to show that these maps induced by elements of V anticommute with each other (see for example [17, A2.4.1]). Thus we get maps $\bigwedge^j V \to \mathrm{Hom}(U^i, U^{i-j})$ which together give a graded ring homomorphism $\bigwedge V \to \mathrm{Hom}(\oplus_i U^i, \oplus_i U^i)$. In fact this construction gives all the homomorphisms between the U^i:

Lemma 4.3. *The maps*

$$\textstyle\bigwedge^j V \to \mathrm{Hom}(U^i, U^{i-j}), \quad 0 \leq i, i-j \leq n\,,$$

described above are isomorphisms. Under these isomorphisms an element $e \in \bigwedge^j V$ acts by contraction on the fibers of the U^i:

$$\begin{tikzcd} \bigwedge^i (V/\langle a \rangle)^* \arrow[r, hook] \arrow[d] & \bigwedge^i W \arrow[d, "e"] \\ \bigwedge^{i-j} (V/\langle a \rangle)^* \arrow[r, hook] & \bigwedge^{i-j} W \ . \end{tikzcd}$$

Proof. Every homomorphism $U^i \to U^{i-j}$ lifts uniquely to a homomorphism between shifted Koszul complexes:

$$\begin{tikzcd} 0 \arrow[r] & \bigwedge^{n+1} W \otimes \mathcal{O}(i-n-1) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigwedge^j W \otimes \mathcal{O}(i-j) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \bigwedge^{n+1-j} W \otimes \mathcal{O}(i-n-1) \arrow[r] & \cdots \arrow[r] & \mathcal{O}(i-j) \arrow[r] & 0 \end{tikzcd}$$

Indeed, the corresponding obstructions vanish by Remarks 4.1 and 4.2. All results follow since the vertical arrows are necessarily given by contraction with an element in

$$\mathrm{Hom}(\textstyle\bigwedge^j W \otimes \mathcal{O}(i-j), \mathcal{O}(i-j)) \cong \textstyle\bigwedge^j V\,. \quad \square$$

In practical terms, these results say that a map $U^i \xrightarrow{e} U^{i-j}$ is represented as

$$\begin{tikzcd}
\bigwedge^{i+1} W \otimes \mathcal{O}(-1) \arrow[r, two heads] \arrow[d, "e"] & U^i \arrow[d] \\
\bigwedge^{i-j+1} W \otimes \mathcal{O}(-1) \arrow[r, two heads] & U^{i-j}
\end{tikzcd}$$

if $0 < i - j \leq i \leq n$, and as the composite

$$\begin{tikzcd}
\bigwedge^{i+1} W \otimes \mathcal{O}(-1) \arrow[r, two heads] & U^i \arrow[r, hook] \arrow[d] & \bigwedge^i W \otimes \mathcal{O} \arrow[d, "e"] \\
& U^0 \arrow[r, phantom, "="] & \mathcal{O}
\end{tikzcd}$$

if $0 = i - j < i \leq n$.

A map from a sum of copies of various U^i to another such sum is given by a homogeneous matrix over the exterior algebra E. In general it is an interesting problem to relate properties of the matrix to properties of the map. Here is one relation which is easy. We will apply it later on in this chapter.

Proposition 4.4. *If*

$$r\,U^i \xrightarrow{B} s\,U^{i-1}$$

is a homomorphism, that is, if B is an $s \times r$-matrix with entries in V, then the following condition is necessary for B to be surjective: If $(b_1, \ldots, b_r)$ is a non-trivial linear combination of the rows of B, then

$$\dim \mathrm{span}(b_1, \ldots, b_r) \geq i + 1.$$

Proof. B is surjective if and only if its dual map is injective on fibers:

$$s \textstyle\bigwedge^{i-1}(V/\langle a \rangle) \xrightarrow{\wedge B^t} r \textstyle\bigwedge^i (V/\langle a \rangle)$$

is injective for any line $\langle a \rangle \subset V$. Consider a non-trivial linear combination $(b_1, \ldots, b_r)^t$ of the columns of B^t, and write $d = \dim\,\mathrm{span}(b_1, \ldots, b_r)$. If $d = i$, then B^t is not injective at any point of $\mathbf{P}(W)$ corresponding to a vector in $\mathrm{span}(b_1, \ldots, b_r)$. If $d < i$, then B^t is not injective at any point of $\mathbf{P}(W)$. $\square$

5 Cohomology and Monads

The technique of monads provides powerful tools for problems such as the construction and classification of coherent sheaves with prescribed invariants. This section is an introduction to monads. We demonstrate their usefulness, which is not obvious at first glance, by reviewing the classification of stable rank 2 vector bundles on the projective plane (see [4], [31], and [26]). Recall that stable bundles admit moduli (see [22], [33], and [34]).

The basic idea behind monads is to represent arbitrary coherent sheaves in terms of simpler sheaves such as line bundles or bundles of differentials, and in terms of homomorphisms between these simpler sheaves. If M is a finitely generated graded S-module, with associated sheaf $\mathcal{F} = \tilde{M}$, then the sheafification of the minimal free resolution of M is a monad for $\mathcal{F}$ which involves direct sums of line bundles and thus homogeneous matrices over S. The Beilinson monad for $\mathcal{F}$, which will be considered in the next section, involves direct sums of twisted bundles of differentials U^i, and thus homogeneous matrices over E.

Definition 5.1. A *monad* on $\mathbf{P}(W)$ is a bounded complex

$$\cdots \longrightarrow \mathcal{K}^{-1} \longrightarrow \mathcal{K}^0 \longrightarrow \mathcal{K}^1 \longrightarrow \cdots$$

of coherent sheaves on $\mathbf{P}(W)$ which is exact except at $\mathcal{K}^0$. The homology $\mathcal{F}$ at $\mathcal{K}^0$ is called the *homology of the monad*, and the monad is said to be a monad for $\mathcal{F}$. We say that the *type of a monad* is determined if the sheaves $\mathcal{K}^i$ are determined. □

There are different ways of representing a given sheaf as the homology of a monad, and the type of the monad depends on the way chosen.

When constructing or classifying sheaves in a given class via monads, one typically proceeds along the following lines.

Step 1. Compute cohomological information which determines the type of the corresponding monads.

Step 2. Construct or classify the differentials of the monads.

There are no general recipes for either step and some cases require sophisticated ideas and quite a bit of intuition (see Example 7.2 below). If one wants to classify, say, vector bundles, then a third step is needed:

Step 3. Determine which monads lead to isomorphic vector bundles.

One of the first successful applications of this approach was the classification of (Gieseker-)stable rank 2 vector bundles with even first Chern class $c_1 \in \mathbf{Z}$ on the complex projective plane by Barth [4], who detected geometric properties of the corresponding moduli spaces without giving an explicit description of the differentials in the second step. The same ideas apply in the case c_1 odd which we are going to survey in what follows (see [31], [26], and [37] for full details and proofs).

In general, rank 2 vector bundles enjoy properties which are not shared by all vector bundles.

Remark 5.2. Every rank 2 vector bundle $\mathcal{F}$ on $\mathbf{P}(W)$ is *self-dual*, that is, it admits a symplectic structure. Indeed, the map

$$\mathcal{F} \otimes \mathcal{F} \xrightarrow{\wedge} \bigwedge^2 \mathcal{F} \cong \mathcal{O}_{\mathbf{P}(W)}(c_1)$$

induces an isomorphism $\varphi : \mathcal{F} \xrightarrow{\cong} \mathcal{F}^*(c_1)$ with $\varphi = -\varphi^*(c_1)$ (here c_1 is the first Chern class of $\mathcal{F}$). In particular there are isomorphisms

$$(\mathrm{H}^j \mathcal{F}(i))^* \cong \mathrm{H}^{n-j} \mathcal{F}(-i-n-1-c_1)$$

by Serre duality. □

We will not give a general definition of stability here. For rank 2 vector bundles stability can be characterized as follows (see [37]).

Remark 5.3. If $\mathcal{F}$ is a rank 2 vector bundle on $\mathbf{P}(W)$, then the following hold:

(1) $\mathcal{F}$ is stable if and only if $\mathrm{Hom}(\mathcal{F}, \mathcal{F}) \cong K$. In this case the symplectic structure on $\mathcal{F}$ is uniquely determined up to scalars.

(2) By tensoring with a line bundle we can *normalize* $\mathcal{F}$ so that its first Chern class is 0 or -1. In this case $\mathcal{F}$ is stable if and only if it has no global sections. □

Example 5.4. By the results of the previous section the twisted cotangent bundle U on the projective plane is a stable rank 2 vector bundle with Chern classes $c_1 = -1$ and $c_2 = 1$. □

Remark 5.5. The generalized theorem of Riemann-Roch yields a polynomial in $\mathbb{Q}[c_1, \ldots, c_r]$ which gives the Euler characteristic $\chi \mathcal{F} = \sum_j (-1)^j \mathrm{h}^j \mathcal{F}$ for every rank r vector bundle $\mathcal{F}$ on $\mathbf{P}(W)$ with Chern classes $c_1, \ldots, c_r$. This polynomial can be determined by interpreting the generalized theorem of Riemann-Roch or by computing the Euler characteristic for enough special bundles of rank r (like direct sums of line bundles). For a rank 2 vector bundle on the projective plane, for example, one obtains

$$\chi(\mathcal{F}) = (c_1^2 - 2c_2 + 3c_1 + 4)/2\,. \quad \square$$

We now focus on stable rank 2 vector bundles on the complex projective plane $\mathbf{P}^2(\mathbf{C}) = \mathbf{P}(W)$ with first Chern class $c_1 = -1$. Let $\mathcal{F}$ be such a bundle.

Remark 5.6. Since $\mathcal{F}$ is stable and normalized its second Chern class c_2 must be ≥ 1. Indeed,

$$\mathrm{H}^2 \mathcal{F}(i-2) = \mathrm{H}^0 \mathcal{F}(-i) = 0 \quad \text{for} \quad i \geq 0$$

by Remarks 5.2 and 5.3, and $\chi(\mathcal{F}(i)) = (i+1)^2 - c_2$ by Riemann-Roch. In particular the dimensions $\mathrm{h}^j\mathcal{F}(i)$ in the range $-2 \le i \le 0$ are as in the following cohomology table (a zero is represented by an empty box):

$j \backslash i$	-2	-1	0
2			
1	$c_2 - 1$	c_2	$c_2 - 1$
0			

$\square$

We abbreviate $\mathcal{O} = \mathcal{O}_{\mathbf{P}^2(\mathbf{C})}$ and go through the three steps above.

Step 1. In this step we show that $\mathcal{F}$ is the homology of a monad of type

$$0 \to \mathrm{H}^1\mathcal{F}(-2) \otimes U^2 \to \mathrm{H}^1\mathcal{F}(-1) \otimes U \to \mathrm{H}^1\mathcal{F} \otimes \mathcal{O} \to 0\ ,$$

where the middle term occurs in cohomological degree 0. This actually follows from the general construction of Beilinson monads presented in the next chapter and the fact that $\mathrm{H}^2\mathcal{F}(i-2) = \mathrm{H}^0\mathcal{F}(-i) = 0$ for $2 \ge i \ge 0$ (see Remark 5.6). Here we derive the existence of the monad directly with Horrocks' technique of killing cohomology [24], which requires further cohomological information. Such information is typically obtained by restricting the given bundles to linear subspaces. In our case we consider the Koszul complex on the equations of a point $p \in \mathbf{P}^2(\mathbf{C})$:

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\binom{-x'}{x}} 2\mathcal{O}(-1) \xrightarrow{(x\ x')} \mathcal{O} \longrightarrow \mathcal{O}_p \longrightarrow 0\ .$$

By tensoring with $\mathcal{F}(i+1)$ and taking cohomology we find that $\mathrm{H}^1\mathcal{F}$ generates $\mathrm{H}^1_{\ge 0}\mathcal{F}$. Indeed, the composite map

$$\big(x\ x'\big) : 2\mathrm{H}^1\mathcal{F}(i) \longrightarrow \mathrm{H}^1(\mathcal{J}_p \otimes \mathcal{F}(i+1)) \longrightarrow \mathrm{H}^1\mathcal{F}(i+1)$$

is surjective if $i \ge -1$. In particular, if $c_2 = 1$, then $\mathrm{H}^1\mathcal{F}(i) = 0$ for $i \ne -1$ (apply Serre duality for the twists ≤ -2), so $\mathcal{F} \cong U$ is the twisted cotangent bundle by Remark 4.2 since both bundles have the same rank and intermediate cohomology.

If $c_2 \ge 2$ then $\mathrm{H}^1\mathcal{F} \ne 0$, and the identity in

$$\mathrm{Hom}(\mathrm{H}^1\mathcal{F}, \mathrm{H}^1\mathcal{F}) \cong \mathrm{Ext}^1(\mathrm{H}^1\mathcal{F} \otimes \mathcal{O}, \mathcal{F})$$

defines an extension

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathrm{H}^1\mathcal{F} \otimes \mathcal{O} \to 0\ ,$$

where $\mathrm{H}^1_{\geq 0}\,\mathcal{G} = 0$, and where $\mathcal{G}$ is a vector bundle (apply Serre's criterion in Section 4). Similarly, by taking Serre duality into account, we obtain an extension

$$0 \to \mathrm{H}^1\mathcal{F}(-2) \otimes U^2 \to \mathcal{H} \to \mathcal{F} \to 0 \ ,$$

where $\mathcal{H}$ is a vector bundle with $\mathrm{H}^1_{\leq -2}\mathcal{H} = 0$. The two extensions fit into a commutative diagram with exact rows and and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^1\mathcal{F}(-2) \otimes U^2 \arrow[r] \arrow[d, phantom, "||"] & \mathcal{H} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^1\mathcal{F}(-2) \otimes U^2 \arrow[r, "\alpha"] & \mathcal{B} \arrow[r] \arrow[d, "\beta"] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\
 & & \mathrm{H}^1\mathcal{F} \otimes \mathcal{O} \arrow[r, phantom, "="] \arrow[d] & \mathrm{H}^1\mathcal{F} \otimes \mathcal{O} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

since, for example, the extension in the top row lifts uniquely to an extension as in the middle row (the obstructions in the corresponding Ext-sequence vanish). Then $\mathcal{B} \cong \mathrm{H}^1\mathcal{F}(-1) \otimes U$ since by construction these bundles have the same rank and intermediate cohomology. What we have is the *display* of (the short exact sequences associated to) a monad

$$0 \longrightarrow \mathrm{H}^1\mathcal{F}(-2) \otimes U^2 \xrightarrow{\alpha} \mathrm{H}^1\mathcal{F}(-1) \otimes U \xrightarrow{\beta} \mathrm{H}^1\mathcal{F} \otimes \mathcal{O} \longrightarrow 0$$

for $\mathcal{F}$.

Step 2. Our task in this step is to describe what maps α and β could be the differentials of a monad as above. In fact we give a description in terms of linear algebra for which it is enough to deal with one of the differentials, say α, since the self-duality of $\mathcal{F}$ and the vanishing of certain obstructions allows one to represent $\mathcal{F}$ as the homology of a "self-dual" monad. Let us abbreviate $A = \mathrm{H}^1\mathcal{F}(-2)$, $B = \mathrm{H}^1\mathcal{F}(-1)$ and $A^* \cong \mathrm{H}^1\mathcal{F}$. By chasing the displays of a monad as above and its dual we see that the symplectic structure on $\mathcal{F}$ lifts to a unique isomorphism of monads

$$\begin{tikzcd}
0 \arrow[r] & A \otimes U^2 \arrow[r, "\alpha"] \arrow[d, "\Phi"] & B \otimes U \arrow[r, "\beta"] \arrow[d, "\Psi"] & A^* \otimes \mathcal{O} \arrow[r] \arrow[d, "-\Phi^*(-1)"] & 0 \\
0 \arrow[r] & A \otimes \mathcal{O}(-1) \arrow[r, "\beta^*(-1)"] & B^* \otimes U^*(-1) \arrow[r, "\alpha^*(-1)"] & A^* \otimes (U^2)^*(-1) \arrow[r] & 0
\end{tikzcd}$$

with $\Psi = -\Psi^*(-1)$. Indeed, the corresponding obstructions vanish (see [5] and [37, II, 4.1] for a discussion of this argument in a general context). Ψ is the tensor product of an isomorphism $q : B \to B^*$ and a symplectic form $\iota \in \mathrm{Hom}(U, U^*(-1)) \cong \mathbf{C}$ on U. Note that q is symmetric since $-(q \otimes \iota) = (q \otimes \iota)^*(-1) = q^* \otimes \iota^*(-1) = -q^* \otimes \iota$. We may and will now assume that $\mathcal{F}$ is the homology of a *self-dual monad*, where self-dual means that $\beta = \alpha^d := \alpha^*(-1) \circ (q \otimes \iota)$. The monad conditions

(α_1) $\alpha^d \circ \alpha = 0$, and

(α_2) α is a vector bundle monomorphism (α^d is an epimorphism)

can be rewritten in terms of linear algebra as follows. The identifications in Lemma 4.3 allow one to view

$$\alpha \in \mathrm{Hom}(A \otimes U^2, B \otimes U) \cong V \otimes \mathrm{Hom}(A, B)$$

as a homomorphism $\alpha : W \to \mathrm{Hom}(A, B)$ operating by $\xi \otimes (x \wedge x') \to \alpha(x)(\xi) \otimes x' - \alpha(x')(\xi) \otimes x$ on the fibers of $A \otimes U^2$. Similarly we consider α^d as the homomorphism $\alpha^d : W \to \mathrm{Hom}(B, A^*), x \mapsto \alpha^*(x) \circ q$, operating by $\eta \otimes x \to \alpha^d(x)(\eta)$ on the fibers of $B \otimes U$. Then

(α_1') $\alpha^d(x) \circ \alpha(x') = \alpha^d(x') \circ \alpha(x)$ for all $x, x' \in W$, and

(α_2') for every $\xi \in A \setminus \{0\}$ the map $W \to B$, $x \to \alpha(x)(\xi)$ has rank ≥ 2.

Example 5.7. If $c_2 = 2$, then the monads can be written (non-canonically) as

$$0 \longrightarrow U^2 \xrightarrow{\binom{a}{b}} 2U \xrightarrow{(a\ b)} \mathcal{O} \longrightarrow 0\ ,$$

where a, b are two vectors in V. In this case (α_1) gives no extra condition and (α_2) means that a and b are linearly independent. If a and b are explicitly given, then we can compute the homology of the monad with the help of *Macaulay 2*:

```
i25 : S = ZZ/32003[x_0..x_2];
```

U is obtained from the Koszul complex resolving $S/(x_0, x_1, x_2)$ by tensoring the cokernel of the differential $\bigwedge^3 W \otimes S(-3) \to \bigwedge^2 W \otimes S(-2)$ with $S(1)$ (and sheafifying).

```
i26 : U = coker koszul(3,vars S) ** S^{1};
```

For representing α and α^d we also need the differential $\bigwedge^2 W \otimes S(-2) \to W \otimes S(-1)$ of the Koszul complex.

```
i27 : k2 = koszul(2,vars S)

o27 = {1} | -x_1 -x_2 0    |
      {1} | x_0  0    -x_2 |
      {1} | 0    x_0  x_1  |

               3       3
o27 : Matrix S  <--- S
```

The expression `koszul(2,vars S)` computes a matrix representing the differential with respect to the monomial bases $x_0 \wedge x_1, x_0 \wedge x_2, x_1 \wedge x_2$ of $\bigwedge^2 W$ and x_0, x_1, x_2 of W. We pick $(a, b) = (e_1, e_2)$ and represent the corresponding maps α and α^d with respect to the monomial bases (see the discussion following Lemma 4.3).

```
i28 : alpha = map(U ++ U, S^{-1}, transpose{{0,-1,0,1,0,0}});

o28 : Matrix

i29 : alphad = map(S^1, U ++ U, matrix{{0,1,0,0,0,1}} * (k2 ++ k2));

o29 : Matrix
```

Prune computes a minimal presentation.

```
i30 : F = prune homology(alphad, alpha);

i31 : betti  F

o31 = relations : total: 3 1
                      1: 2 .
                      2: 1 1
```

In the next section we will present a more elegant way of computing the homology of Beilinson monads. □

We go back to the general case and reverse our construction. Let A and B be $\mathbf{C}$-vector spaces of the appropriate dimensions, let q be a non-degenerate quadratic form on B, and let

$$\widetilde{\mathcal{M}} = \{\alpha \in \mathrm{Hom}(W, \mathrm{Hom}(A, B)) \mid \alpha \text{ satisfies } (\alpha_1') \text{ and } (\alpha_2')\} \ .$$

Then every $\alpha \in \widetilde{\mathcal{M}}$ defines a self-dual monad as above whose homology is a stable rank 2 vector bundle on $\mathbf{P}^2(\mathbf{C})$ with Chern classes $c_1 = -1$ and c_2. In this way we obtain a description of the differentials of the monads which is not as explicit as we might have hoped (with the exception of the case $c_2 = 2$). It is, however, enough for detecting geometric properties of the corresponding moduli spaces.

Step 3. Constructing the moduli spaces means to parametrize the isomorphism classes of our bundles in a convenient way. We very roughly outline how to do that. Let $\mathrm{O}(B)$ be the orthogonal group of (B, q), and let $G := \mathrm{GL}(A) \times \mathrm{O}(B)$. Then G acts on $\widetilde{\mathcal{M}}$ by $((\Phi, \Psi), \alpha) \mapsto \Psi\alpha\Phi^{-1}$, where $\Psi\alpha\Phi^{-1}(x) := \Psi\alpha(x)\Phi^{-1}$. We may consider an element $(\Phi, \Psi) \in G$ as an isomorphism between the monad defined by α and the monad defined by $\Psi\alpha\Phi^{-1}$. By going back and forth between isomorphisms of bundles and isomorphisms of monads one shows that the stabilizer of G in each point is $\{\pm 1\}$, and that our construction induces a bijection between the set of isomorphism classes of stable rank 2 vector bundles on $\mathbf{P}^2(\mathbf{C})$ with Chern classes $c_1 = -1$ and c_2 and $\mathcal{M} := \widetilde{\mathcal{M}}/G_0$, where $G_0 := G/\{\pm 1\}$. With the help of a universal monad over $\mathbf{P}^2(\mathbf{C}) \times \widetilde{\mathcal{M}}$ one proves that the analytic structure on $\widetilde{\mathcal{M}}$ descends to an analytic structure on $\mathcal{M}$ so that $\mathcal{M}$ is smooth of dimension $\mathrm{h}^1\mathcal{F}^* \otimes \mathcal{F} = 4c_2 - 4$ in each point (the obstructions for smoothness in the point corresponding to $\mathcal{F}$ lie in $\mathrm{H}^2\mathcal{F}^* \otimes \mathcal{F}$ which is zero). Moreover the homology of the universal monad tensored by a suitable line bundle descends to a universal family over $\mathcal{M}$ (here one needs $c_1 = -1$). In other words, $\mathcal{M}$ is what one calls a fine moduli space for our bundles. Further efforts show that $\mathcal{M}$ is irreducible and rational.

Remark 5.8. Horrocks' technique of killing cohomology always yields 3-term monads. In general, the bundle in the middle can be pretty complicated. $\square$

6 The Beilinson Monad

We can use the Tate resolution associated to a sheaf to give a construction of a complex first described by Beilinson [6], which gives a powerful method for deriving information about a sheaf from information about a few of its cohomology groups. The general idea is the following:

Suppose that $\mathcal{A}$ is an additive category and consider a graded object $\oplus_{i=0}^{n+1} U^i$ in $\mathcal{A}$. Given a graded ring homomorphism $E \to \mathrm{End}_{\mathcal{A}}(\oplus_{i=0}^{n+1} U^i)$ we can make an additive functor from the category of free E-modules to $\mathcal{A}$: On objects we take

$$\omega_E(i) \mapsto \begin{cases} U^i & \text{for } 0 \le i \le n+1 \text{ and;} \\ 0 & \text{otherwise.} \end{cases}$$

To define the functor on maps, we use

$$\begin{aligned} \mathrm{Hom}_E(\omega_E(i), \omega_E(j)) &= \mathrm{Hom}_E(E(i), E(j)) \\ &= E_{j-i} \longrightarrow \mathrm{End}(\oplus U^i)_{j-i} \longrightarrow \mathrm{Hom}(U^i, U^j)\ . \end{aligned}$$

(Note that we could have taken any twist of E in place of $\omega_E \cong E(-n-1)$; the choice of ω_E is made to simplify the statement of Theorem 6.1, below.)

We shall be interested in the special case where $\mathcal{A}$ is the category of coherent sheaves on $\mathbf{P}(W)$ and where $U^i = \Omega^i_{\mathbf{P}(W)}(i)$ as in Section 4. Further examples may be obtained by taking U^i to be the i^{th} exterior power of the tautological subbundle U_k on the Grassmannian of k-planes in W for any k; the case we have taken here is the case $k = n$. See [19] for more information on the general case and applications to the computation of resultants and more general Chow forms.

Applying the functor just defined to the Tate resolution $\mathbf{T}(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on $\mathbf{P}(W)$, and using Theorem 3.1, we get a complex

$$\Omega(\mathcal{F}): \quad \cdots \longrightarrow \oplus_j \mathrm{H}^j\mathcal{F}(i-j) \otimes U^{j-i} \longrightarrow \ \ldots,$$

where the term we have written down occurs in cohomological degree i. The resolution $\mathbf{T}(\mathcal{F})$ is well-defined up to homotopy, so the same is true of $\Omega(\mathcal{F})$. Since $U^k = 0$ unless $0 \le k \le n$ the only cohomology groups of $\mathcal{F}$ that are actually involved in $\Omega(\mathcal{F})$ are $\mathrm{H}^j\mathcal{F}(k)$ with $-n \le k \le 0$; $\Omega(\mathcal{F})$ is of type

$$\begin{array}{ccccccccc}
0 \to & \mathrm{H}^0\mathcal{F}(-n)\otimes U^n & \to \cdots \to & \oplus_{j=0}^n \mathrm{H}^j\mathcal{F}(-j)\otimes U^j & \to \cdots \to & \mathrm{H}^n\mathcal{F}\otimes U^0 & \to 0 \\
 & \| & & \| & & \| & \\
0 \longrightarrow & \Omega^{-n}(\mathcal{F}) & \longrightarrow \cdots \longrightarrow & \Omega^0(\mathcal{F}) & \longrightarrow \cdots \longrightarrow & \Omega^n(\mathcal{F}) & \longrightarrow 0 \quad .
\end{array}$$

For applications it is important to note that instead of working with $\Omega(\mathcal{F})$ one can also work with $\Omega(\mathcal{F}(i))$ for some twist i. This gives one some freedom in choosing the cohomology groups of $\mathcal{F}$ to be involved.

To see a simple example, consider again the structure sheaf $\mathcal{O}_p$ of the subvariety consisting of a point $p \in \mathbf{P}(W)$. Write I for the homogeneous ideal of p, and let $a \in V = W^*$ be a non-zero functional vanishing on the linear forms in I as before. The Tate resolution of the homogeneous coordinate ring S/I has already been computed, and we have seen that it depends only on the sheaf $\widetilde{S/I} = \mathcal{O}_p$. From the computation of $\mathbf{T}(S/I) = \mathbf{T}(\mathcal{O}_p)$ made in Section 3 we see that $\Omega(\mathcal{O}_p)$ takes the form

$$\Omega(\mathcal{O}_p): \quad 0 \to U^n \xrightarrow{\ a\ } U^{n-1} \xrightarrow{\ a\ } \cdots \xrightarrow{\ a\ } U^1 \xrightarrow{\ a\ } U^0 \longrightarrow 0\ ,$$

with U^i in cohomological degree $-i$.

We have already noted that the map $a : U = U^1 \longrightarrow U^0 = \mathcal{O}_{\mathbf{P}(W)}$ is the composite of the tautological embedding $U \subset W \otimes \mathcal{O}_{\mathbf{P}(W)}$ with the map $a \otimes 1 :\ W \otimes \mathcal{O}_{\mathbf{P}(W)} \to \mathcal{O}_{\mathbf{P}(W)}$. Thus the image of $a : U^1 \to \mathcal{O}_{\mathbf{P}(W)}$ is the ideal sheaf of p, and we see that the homology of the complex $\Omega(\mathcal{O}_p)$ at U^0 is $\mathcal{O}_p$. One can check further that $\Omega(\mathcal{O}_p)$ is the Koszul complex associated with the map $a : U^1 \to \mathcal{O}_{\mathbf{P}(W)}$, and it follows that the homology of $\Omega(\mathcal{O}_p)$ at U^i is 0 for $i > 0$. The following result shows that this is typical.

Theorem 6.1 ([18]). *If $\mathcal{F}$ is a coherent sheaf on $\mathbf{P}(W)$, then the only non-vanishing homology of the complex $\Omega(\mathcal{F})$ is*

$$\mathrm{H}^0(\Omega(\mathcal{F})) = \mathcal{F}\,. \qquad \square$$

The existence of a complex satisfying the theorem and having the same terms as $\Omega(\mathcal{F})$ was first asserted by Beilinson in [6], and thus we will call $\Omega(\mathcal{F})$ a *Beilinson monad* for $\mathcal{F}$. Existence proofs via a somewhat less effective construction than the one given here may be found in [28] and [2].

The explicitness of the construction via Tate resolutions allows one to detect properties of the differentials of Beilinson monads. Let us write

$$\begin{aligned}
d_{ij}^{(r)} &\in \mathrm{Hom}(\mathrm{H}^j\mathcal{F}(i-j)\otimes U^{j-i}, \mathrm{H}^{j-r+1}\mathcal{F}(i-j+r)\otimes U^{j-i-r}) \\
&\cong \textstyle\bigwedge^r V \otimes \mathrm{Hom}(\mathrm{H}^j\mathcal{F}(i-j), \mathrm{H}^{j-r+1}\mathcal{F}(i-j+r)) \\
&\cong \mathrm{Hom}(\textstyle\bigwedge^r W \otimes \mathrm{H}^j\mathcal{F}(i-j), \mathrm{H}^{j-r+1}\mathcal{F}(i-j+r))
\end{aligned}$$

for the degree r maps actually occurring in $\Omega(\mathcal{F})$.

Remark 6.2. The constant maps $d_{ij}^{(0)}$ in $\Omega(\mathcal{F})$ are zero since $\mathbf{T}(\mathcal{F})$ is minimal. $\square$

Proposition 6.3 ([18]). *The linear maps $d_{ij}^{(1)}$ in $\Omega(\mathcal{F})$ correspond to the multiplication maps*

$$W \otimes \mathrm{H}^j\mathcal{F}(i-j) \to \mathrm{H}^j\mathcal{F}(i-j+1)\,. \qquad \square$$

This follows from the identification of the linear strands in $\mathbf{T}(\mathcal{F})$ (see the discussion following Theorem 3.1). The higher degree maps in $\mathbf{T}(\mathcal{F})$ and $\Omega(\mathcal{F})$, however, are not yet well-understood.

Since $(\mathbf{T}(\mathcal{F}))[1] = \mathbf{T}(\mathcal{F}(1))$ we can compare the differentials in $\Omega(\mathcal{F})$ with those in $\Omega(\mathcal{F}(1))$:

Proposition 6.4 ([18]). *If the maps $d_{ij}^{(r)}$ in $\Omega(\mathcal{F})$ and $d_{i-1,j}^{(r)}$ in $\Omega(\mathcal{F}(1))$ both actually occur, then they correspond to the same element in*

$$\bigwedge^r V \otimes \mathrm{Hom}(\mathrm{H}^j\mathcal{F}(i-j), \mathrm{H}^{j-r+1}\mathcal{F}(i-j+r))\ . \qquad \square$$

In what follows we present some *Macaulay 2* code for computing Beilinson monads. Our functions `sortedBasis`, `beilinson1`, `U`, and `beilinson` reflect what we did in Example 5.7 .

The expression `sortedBasis(i,E)` sorts the monomials of degree i in E to match the order of the columns of `koszul(i,vars S)`, where our conventions with respect to S and E are as in Section 2, and where we suppose that the monomial order on E is reverse lexicographic, the *Macaulay 2* default order.

```
i32 : sortedBasis = (i,E) -> (
           m := basis(i,E);
           p := sortColumns(m,MonomialOrder=>Descending);
           m_p);
```

For example:

```
i33 : S=ZZ/32003[x_0..x_3];

i34 : E=ZZ/32003[e_0..e_3,SkewCommutative=>true];

i35 : koszul(2,vars S)

o35 = {1} | -x_1 -x_2 0    -x_3 0    0    |
      {1} | x_0  0    -x_2 0    -x_3 0    |
      {1} | 0    x_0  x_1  0    0    -x_3 |
      {1} | 0    0    0    x_0  x_1  x_2  |

               4       6
o35 : Matrix S  <--- S
```

```
i36 : sortedBasis(2,E)

o36 = | e_0e_1 e_0e_2 e_1e_2 e_0e_3 e_1e_3 e_2e_3 |

              1       6
o36 : Matrix E  <--- E
```

If $e \in E$ is homogeneous of degree j, then `beilinson1(e,j,i,S)` computes the map $U^i \xrightarrow{e} U^{i-j}$ on $\mathbf{P}^n = \operatorname{Proj} S$. If $0 < i - j \le i \le n$, then the result is a matrix representing the map $\bigwedge^{i+1} W \otimes S(-1) \xrightarrow{e \otimes 1} \bigwedge^{i-j+1} W \otimes S(-1)$ defined by contraction with e. If $0 = i-j < i \le n$, then the result is a matrix representing the composite of the map $\bigwedge^{i} W \otimes S \xrightarrow{e \otimes 1} S$ with the Koszul differential $\bigwedge^{i+1} W \otimes S(-1) \to \bigwedge^{i} W \otimes S$. Note that the degrees of the result are not set correctly since the functions `U` and `beilinson` below are supposed to do that.

```
i37 : beilinson1=(e,dege,i,S)->(
           E := ring e;
           mi := if i < 0 or i >= numgens E then map(E^1, E^0, 0)
                 else if i === 0 then id_(E^1)
                 else sortedBasis(i+1,E);
           r := i - dege;
           mr := if r < 0 or r >= numgens E then map(E^1, E^0, 0)
                 else sortedBasis(r+1,E);
           s = numgens source mr;
           if i === 0 and r === 0 then
                substitute(map(E^1,E^1,{{e}}),S)
           else if i>0 and r === i then substitute(e*id_(E^s),S)
           else if i > 0 and r === 0 then
                (vars S) * substitute(contract(diff(e,mi),transpose mr),S)
           else substitute(contract(diff(e,mi), transpose mr),S));
```

For example:

```
i38 : beilinson1(e_1,1,3,S)

o38 = {-3} | 0 |
      {-3} | 0 |
      {-3} | 1 |
      {-3} | 0 |

              4       1
o38 : Matrix S  <--- S

i39 : beilinson1(e_1,1,2,S)

o39 = {-2} | 0  0  0 0 |
      {-2} | -1 0  0 0 |
      {-2} | 0  0  0 0 |
      {-2} | 0  -1 0 0 |
      {-2} | 0  0  0 0 |
      {-2} | 0  0  0 1 |

              6       4
o39 : Matrix S  <--- S

i40 : beilinson1(e_1,1,1,S)

o40 = | x_0 0 -x_2 0 -x_3 0 |
```

```
               1       6
o40 : Matrix S  <--- S
```

The function `U` computes the bundles U^i on $\operatorname{Proj} S$:

```
i41 : U = (i,S) -> (
           if i < 0 or i >= numgens S then S^0
           else if i === 0 then S^1
           else cokernel koszul(i+2,vars S) ** S^{i});
```

Finally, if $o : \oplus E(-a_i) \to \oplus E(-b_j)$ is a homogeneous matrix over E, then `beilinson(o,S)` computes the corresponding map $o : \oplus U^{a_i} \to \oplus U^{b_j}$ on $\operatorname{Proj} S$ by calling `beilinson1` and `U`.

```
i42 : beilinson = (o,S) -> (
           coldegs := degrees source o;
           rowdegs := degrees target o;
           mats = table(numgens target o, numgens source o,
                    (r,c) -> (
                         rdeg = first rowdegs#r;
                         cdeg = first coldegs#c;
                         overS = beilinson1(o_(r,c),cdeg-rdeg,cdeg,S);
                         -- overS = substitute(overE,S);
                         map(U(rdeg,S),U(cdeg,S),overS)));
           if #mats === 0 then matrix(S,{{}})
           else matrix(mats));
```

With these functions the code in Example 5.7 can be rewritten as follows:

```
i43 : S=ZZ/32003[x_0..x_2];

i44 : E = ZZ/32003[e_0..e_2,SkewCommutative=>true];

i45 : alphad = map(E^1,E^{-1,-1},{{e_1,e_2}})

o45 = | e_1 e_2 |

               1       2
o45 : Matrix E  <--- E

i46 : alpha = map(E^{-1,-1},E^{-2},{{e_1},{e_2}})

o46 = {1} | e_1 |
      {1} | e_2 |

               2       1
o46 : Matrix E  <--- E

i47 : alphad=beilinson(alphad,S);

o47 : Matrix

i48 : alpha=beilinson(alpha,S);

o48 : Matrix

i49 : F = prune homology(alphad,alpha);

i50 : betti  F

o50 = relations : total: 3 1
                      1: 2 .
                      2: 1 1
```

7 Examples

In this section we give two examples of explicit constructions of Beilinson monads over $\mathbf{P}^4(\mathbf{C}) = \mathbf{P}(W)$ and of classification results based on these monads. As in Section 5 we proceed in three steps. Let us write $\mathcal{O} = \mathcal{O}_{\mathbf{P}^4(\mathbf{C})}$.

Example 7.1. Our first example is taken from the classification of *conic bundles* in $\mathbf{P}^4(\mathbf{C})$, that is, of smooth surfaces $X \subset \mathbf{P}^4(\mathbf{C})$ which are ruled in conics in the sense that there exists a surjective morphism $\pi : X \to C$ onto a smooth curve C such that the general fiber of π is a smooth conic in the given embedding of X. There are precisely three families of such surfaces (see [20] and [9]). Two families, the Del Pezzo surfaces of degree 4 and the Castelnuovo surfaces, are classical. The third family, consisting of *elliptic conic bundles* (conic bundles over an elliptic curve) of degree 8, had been falsely ruled out in two classification papers in the 1980's (see [36] and [27]). Only recently Abo, Decker, and Sasakura [1] constructed and classified such surfaces by considering the Beilinson monads for the suitably twisted ideal sheaves of the surfaces. Let us explain how this works.

Step 1. In this step we suppose that an elliptic conic bundle X as above exists, and we determine the type of the Beilinson monad for the suitably twisted ideal sheaf $\mathcal{J}_X$. We know from the classification of smooth surfaces in $\mathbf{P}^4(\mathbf{C})$ which are contained in a cubic hypersurface (see [38] and [3]) that $\mathrm{H}^0 \mathcal{J}_X(i) = 0$ for $i \leq 3$. It follows from general results such as the theorem of Riemann-Roch that the dimensions $\mathrm{h}^j \mathcal{J}_X(i)$ in range $-2 \leq i \leq 3$ are as follows (here, again, a zero is represented by an empty box):

j \ i	-2	-1	0	1	2	3
4						
3	8	4				
2			1	1	a	b
1					$a+1$	$b+1$
0						

with $a := \mathrm{h}^2 \mathcal{J}_X(2)$ and $b := \mathrm{h}^2 \mathcal{J}_X(3)$ still to be determined. The Beilinson monad for $\mathcal{J}_X(2)$ is thus of type

$$0 \to 8\mathcal{O}(-1) \to 4U^3 \oplus U^2 \to U \oplus (a+1)\mathcal{O} \to a\mathcal{O} \to 0 \ ,$$

where $(a+1)\mathcal{O} \to a\mathcal{O}$ is the zero map (see Remark 6.2), and where consequently U is mapped surjectively onto $a\mathcal{O}$. By Proposition 4.4 this is only possible if $a = 0$. The same idea applied to $\mathcal{J}_X(3)$ shows that then also $b = 0$.

The cohomological information obtained so far determines the type of the Beilinson monad for $\mathcal{J}_X(2)$ and for $\mathcal{J}_X(3)$. We decide to concentrate on the monad for $\mathcal{J}_X(3)$ since its differentials are smaller in size than those of the monad for $\mathcal{J}_X(2)$. In order to ease our calculations further we kill the 4-dimensional space $\mathrm{H}^3\mathcal{J}_X(-1)$. Let us write ω_X for the dualizing sheaf of X. Serre duality on $\mathbf{P}^4(\mathbf{C})$ respectively on X yields canonical isomorphisms

$$\begin{aligned} Z &:= \mathrm{Ext}^1(\mathcal{J}_X(-1), \mathcal{O}(-5)) \\ &\cong (\mathrm{H}^3\mathcal{J}_X(-1))^* \cong (\mathrm{H}^2\mathcal{O}_X(-1))^* \cong \mathrm{H}^0(\omega_X(1)) \ . \end{aligned}$$

The identity in

$$\mathrm{Hom}(Z, Z) \cong \mathrm{Ext}^1(\mathcal{J}_X(-1), Z^* \otimes \mathcal{O}(-5))$$

defines an extension which, twisted by 4, can be written as

$$0 \to 4\mathcal{O}(-1) \to \mathcal{G} \to \mathcal{J}_X(3) \to 0 \ .$$

Let us show that $\mathcal{G}$ is a vector bundle. We know from the classification of scrolls in $\mathbf{P}^4(\mathbf{C})$ (see [30] and [3]) that X is not a scroll. Hence adjunction theory implies that $\omega_X(1)$ is generated by the adjoint linear system $\mathrm{H}^0(\omega_X(1))$ (see [7, Corollary 9.2.2]). It follows by Serre's criterion ([39], see also [35, Theorem 2.2]) that $\mathcal{G}$ is locally free. By construction $\mathcal{G}$ has a cohomology table as follows:

j \ i	-4	-3	-2	-1	0
4					
3					
2		1	1		
1				1	1
0					

So the Beilinson monad of $\mathcal{G}$ is of type

$$0 \to U^3 \xrightarrow{\alpha} U^2 \oplus U \xrightarrow{\beta} \mathcal{O} \to 0 \ .$$

Step 2. Now we proceed the other way around. We show that a rank 5 bundle $\mathcal{G}$ as in the first step exists, and that the dependency locus of four general sections of $\mathcal{G}(1)$ is a surface of the desired type. Differentials which define a monad as above with a locally free homology can be easily found. By Lemma 4.3 α corresponds to a pair of vectors $\alpha = (\alpha_1, \alpha_2)^t \in V \oplus \bigwedge^2 V$. By dualizing (see Remark 4.1) we find that it is a vector bundle monomorphism if and only if $U^2 \oplus U^3 \xrightarrow{\alpha^t} U^1$ is an epimorphism. Equivalently, α_1 is non-zero and α_2 considered as a vector in $\bigwedge^2(V/\langle\alpha_1\rangle)$ is indecomposable (argue as in the proof of Proposition 4.4). Taking the other monad conditions into account we see that we may pick

$$\alpha = \begin{pmatrix} e_4 \\ e_0 \wedge e_2 + e_1 \wedge e_3 \end{pmatrix}$$

and

$$\beta = \left(e_0 \wedge e_2 + e_1 \wedge e_3\, ,\, -e_4\right),$$

where $e_0, \ldots, e_4$ is a basis of V, and that up to isomorphisms of monads and up to the choice of the basis this is the only possibility. We fix $\mathcal{G}$ as the homology of this monad and compute the syzygies of $\mathcal{G}$ with *Macaulay 2*.

```
i51 : S = ZZ/32003[x_0..x_4];

i52 : E = ZZ/32003[e_0..e_4,SkewCommutative=>true];

i53 : beta=map(E^1,E^{-2,-1},{{e_0*e_2+e_1*e_3,-e_4}})

o53 = | e_0e_2+e_1e_3 -e_4 |

               1       2
o53 : Matrix E  <--- E

i54 : alpha=map(E^{-2,-1},E^{-3},{{e_4},{e_0*e_2+e_1*e_3}})

o54 = {2} | e_4           |
      {1} | e_0e_2+e_1e_3 |

               2       1
o54 : Matrix E  <--- E

i55 : beta=beilinson(beta,S);

o55 : Matrix

i56 : alpha=beilinson(alpha,S);

o56 : Matrix

i57 : G = prune homology(beta,alpha);

i58 : betti res G

o58 = total: 10 9 5 1
          1: 10 4 1 .
          2:  . 5 4 1
```

We see in particular that $\mathcal{G}(1)$ is globally generated. Hence the dependency locus of four general sections of $\mathcal{G}(1)$ is indeed a smooth surface in $\mathbf{P}^4(\mathbf{C})$ by Kleiman's Bertini-type result [29]. The smoothness can also be checked with *Macaulay 2* in an example via the built-in Jacobian criterion (see [15] for a speedier method).

```
i59 : foursect = random(S^4, S^10) * presentation G;

                 4       9
o59 : Matrix S  <--- S
```

The function `trim` computes a minimal presentation.

```
i60 : IX = trim minors(4,foursect);

o60 : Ideal of S

i61 : codim IX

o61 = 2

i62 : degree IX

o62 = 8

i63 : codim singularLocus IX

o63 = 5
```

By construction X has the correct invariants and is in fact an elliptic conic bundle as claimed: Since the adjoint linear system $\mathrm{H}^0(\omega_X(1))$ is base point free and 4-dimensional by what has been said in the first step, the corresponding adjunction map $X \to \mathbf{P}^3$ is a morphism which exhibits, as is easy to see, X as a conic bundle over a smooth elliptic curve in $\mathbf{P}^3$ (see [1, Proposition 2.1]).

Step 3. Our discussion in the previous steps gives also a classification result. Up to projectivities the elliptic conic bundles of degree 8 in $\mathbf{P}^4(\mathbf{C})$ are precisely the smooth surfaces arising as the dependency locus of four sections of the bundle $\mathcal{G}(1)$ fixed in Step 2. $\square$

Example 7.2. This example is concerned with the construction and classification of abelian surfaces in $\mathbf{P}^4(\mathbf{C})$, and with the closely related Horrocks-Mumford bundles [25].

Step 1. Horrocks and Mumford found evidence for the existence of a family of abelian surfaces in $\mathbf{P}^4(\mathbf{C})$. Suppose that such a surface X exists. Then the dualizing sheaf of X is trivial, $\omega_X \cong \mathcal{O}_X$, and X has degree 10 (see [21, Example 3.2.15]). The same arguments as in Example 7.1 show that X arises as the zero scheme of a section of a rank 2 vector bundle: There is an extension

$$0 \to \mathcal{O} \to \mathcal{F}(3) \to \mathcal{J}_X(5) \to 0 \ ,$$

where $\mathcal{F}(3)$ is a rank 2 vector bundle with Chern classes $c_1 = 5$ and $c_2 = \deg X = 10$, and where $\mathcal{F}$ has a cohomology table as displayed in Figure 1. In particular $\mathcal{F}$, which has Chern classes $c_1 = -1$ and $c_2 = 4$, is stable by Remark 5.3. A discussion as in Section 5 shows that the Beilinson monad for $\mathcal{F}$ is of type

$$0 \longrightarrow A \otimes \mathcal{O}(-1) \xrightarrow{\alpha} B \otimes U^2 \xrightarrow{\alpha^d} A^* \otimes \mathcal{O} \longrightarrow 0 \ \ ,$$

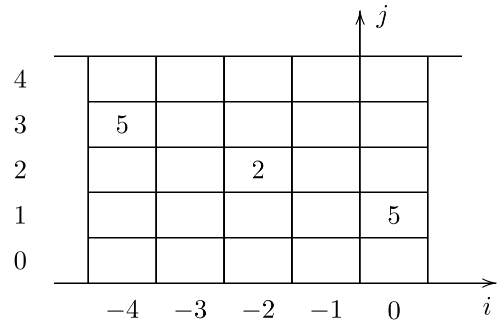

Fig. 1.

with **C**-vector spaces A and B of dimension 5 and 2 respectively, and with $\alpha^d = \alpha^*(-1) \circ (q \otimes \iota)$, where q is a symplectic form on B, and where $\iota : U^2 \xrightarrow{\cong} (U^2)^*(-1)$ is induced by the pairing $U^2 \otimes U^2 \xrightarrow{\wedge} U^4 \cong \mathcal{O}(-1)$. By choosing appropriate bases of A and B we may suppose that α is a 2×5 matrix with entries in $\bigwedge^2 V$ and that $\alpha^d = \alpha^t \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Step 2. As in Example 7.1 we now proceed the other way around. But this time it is not obvious how to define α. Horrocks and Mumford remark that up to projectivities one may suppose that the abelian surfaces in $\mathbf{P}^4(\mathbf{C})$ are invariant under the action of the Heisenberg group H_5 in its Schrödinger representation, and they use the representation theory of H_5 and its normalizer N_5 in $\mathrm{SL}(5, \mathbf{C})$ to find

$$\alpha = \begin{pmatrix} e_2 \wedge e_3 & e_3 \wedge e_4 & e_4 \wedge e_0 & e_0 \wedge e_1 & e_1 \wedge e_2 \\ e_1 \wedge e_4 & e_2 \wedge e_0 & e_3 \wedge e_1 & e_4 \wedge e_2 & e_0 \wedge e_3 \end{pmatrix} ,$$

where $e_0, \ldots, e_4$ is a basis of V. A straightforward computation shows that with this α the desired monad conditions are indeed satisfied. The resulting Horrocks-Mumford bundle $\mathcal{F}_{\mathrm{HM}}$ on $\mathbf{P}^4(\mathbf{C})$ is essentially the only rank 2 vector bundle known on $\mathbf{P}^n(\mathbf{C})$, $n \geq 4$, which does not split as direct sum of two line bundles. Let us compute the syzygies of $\mathcal{F}_{\mathrm{HM}}$ with *Macaulay 2*.

```
i64 : alphad = matrix{{e_4*e_1, e_2*e_3},{e_0*e_2, e_3*e_4},
                      {e_1*e_3, e_4*e_0},{e_2*e_4, e_0*e_1},
                      {e_3*e_0, e_1*e_2}};

                    5       2
o64 : Matrix E  <--- E

i65 : alphad=map(E^5,E^{-2,-2},alphad)

o65 = | -e_1e_4 e_2e_3  |
      | e_0e_2  e_3e_4  |
      | e_1e_3  -e_0e_4 |
      | e_2e_4  e_0e_1  |
      | -e_0e_3 e_1e_2  |
```

```
              5       2
o65 : Matrix E  <--- E

i66 : alpha=syz alphad

o66 = {2} | e_2e_3 e_0e_4 e_1e_2 -e_3e_4 e_0e_1  |
      {2} | e_1e_4 e_1e_3 e_0e_3 e_0e_2  -e_2e_4 |

              2       5
o66 : Matrix E  <--- E

i67 : alphad=beilinson(alphad,S);

o67 : Matrix

i68 : alpha=beilinson(alpha,S);

o68 : Matrix

i69 : FHM = prune homology(alphad,alpha);

i70 : betti res FHM

o70 = total: 19 35 20 2
          3:  4  .  . .
          4: 15 35 20 .
          5:  .  .  . 2

i71 : regularity FHM

o71 = 5

i72 : betti sheafCohomology(presentation FHM,E,-6,6)

o72 = total: 210 100 37 14 10 5 2 5 10 14 37 100 210
         -6: 210 100 35  4  . . . . .  .  .   .   .
         -5:   .   .  2 10 10 5 . . .  .  .   .   .
         -4:   .   .  .  .  . . 2 . .  .  .   .   .
         -3:   .   .  .  .  . . . 5 10 10  2   .   .
         -2:   .   .  .  .  . . . .  .  4 35 100 210
```

Since $\mathrm{H}^0\mathcal{F}_{\mathrm{HM}}(i) = 0$ for $i < 3$ every non-zero section of $\mathcal{F}_{\mathrm{HM}}(3)$ vanishes along a surface (with the desired invariants). Horrocks and Mumford need an extra argument to show that the general such surface is smooth (and thus abelian) since Kleiman's Bertini-type result does not apply ($\mathcal{F}_{\mathrm{HM}}(3)$ is not globally generated). Our explicit construction allows one again to check the smoothness with *Macaulay 2* in an example.

```
i73 : sect =  map(S^1,S^15,0) | random(S^1, S^4);

              1       19
o73 : Matrix S  <--- S
```

We compute the equations of X via a mapping cone.

```
i74 : mapcone = sect || transpose presentation FHM;

              36       19
o74 : Matrix S   <--- S

i75 : fmapcone = res coker mapcone;
```

```
i76 : IX =  trim ideal fmapcone.dd_2;

o76 : Ideal of S

i77 : codim IX

o77 = 2

i78 : degree IX

o78 = 10

i79 : codim singularLocus IX

o79 = 5
```

Step 3. Horrocks and Mumford showed that up to projectivities every abelian surface in $\mathbf{P}^4(\mathbf{C})$ arises as the zero scheme of a section of $\mathcal{F}_{\mathrm{HM}}(3)$. In fact, one can show much more. By a careful analysis of possible Beilinson monads and their restrictions to various linear subspaces Decker [12] proved that every stable rank 2 vector bundle $\mathcal{F}$ on $\mathbf{P}^4(\mathbf{C})$ with Chern classes $c_1 = -1$ and $c_2 = 4$ is the homology of a monad of the type as in Step 1. From geometric properties of the "variety of unstable planes" of $\mathcal{F}$ Decker and Schreyer [14] deduced that up to isomorphisms and projectivities the differentials of the monad coincide with those of $\mathcal{F}_{\mathrm{HM}}$. Together with results from [11] this implies that the moduli space of our bundles is isomorphic to the homogeneous space $\mathrm{SL}(5, \mathbf{C})/N_5$. □

References

1. H. Abo, W. Decker, and N. Sasakura: An elliptic conic bundle on $\mathbf{P}^4$ arising from a stable rank-3 vector bundle. *Math. Z.*, 229:725–741, 1998.
2. V. Ancona and G. Ottaviani: An introduction to the derived categories and the theorem of Beilinson. *Atti Accad. Peloritana Pericolanti Cl. Sci. Fis. Mat. Natur.*, 67:99–110 (1991), 1989.
3. A. Aure: On surfaces in projective 4-space. PhD Thesis, Oslo, 1987.
4. W. Barth: Moduli of vector bundles on the projective plane. *Invent. Math.*, 42:63–91, 1977.
5. W. Barth and K. Hulek: Monads and moduli of vector bundles. *Manuscripta Math.*, 25:323–347, 1978.
6. A. A. Beĭlinson: Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra. *Functional Anal. Appl.*, 12:214–216, 1978.
7. M.C. Beltrametti and A. Sommese: *The adjunction theory of complex projective varieties.* de Gruyter, Berlin, 1995.
8. I. N. Bernšteĭn, I. M. Gel'fand, and S. I. Gel'fand: Algebraic vector bundles on $\mathbf{P}^n$ and problems of linear algebra. *Functional Anal. Appl.*, 12:212–214, 1978.
9. R. Braun and K. Ranestad: Conic bundles in projective fourspace. In P. Newstead, editor: , *Algebraic geometry. Papers presented for the EUROPROJ conferences held in Catania, Italy, September 1993, and Barcelona, Spain, September 1994*, pages 331–339. Marcel Dekker, New York, 1994.

10. R.-O. Buchweitz: Appendix to Cohen-Macaulay modules on quadrics. In *Singularities, representation of algebras, and vector bundles (Lambrecht, 1985)*, pages 96–116. Springer, Berlin, 1987.
11. W. Decker: Das Horrocks-Mumford-Bündel und das Modul-Schema für stabile 2-Vektorbündel über $\mathbf{P}_4$ mit c_1=-1, c_2=4. *Math. Z.*, 188:101–110, 1984.
12. W. Decker: Stable rank 2 vector bundles with Chern-classes c_1=-1, c_2=4. *Math. Ann.*, 275:481–500, 1986.
13. W. Decker: Monads and cohomoloy modules of rank 2 vector bundles. *Compositio Math.*, 76:7–17, 1990.
14. W. Decker and F.-O. Schreyer: On the uniqueness of the Horrocks-Mumford-bundle. *Math. Ann.*, 273:415–443, 1986.
15. W. Decker and F.-O. Schreyer: Non-general type surfaces in $\mathbf{P}^4$: Some remarks on bounds and constructions. *J. Symbolic Computation*, 29:545–582, 2000.
16. D. Eisenbud: Computing cohomology. A chapter in [41].
17. D. Eisenbud: *Commutative algebra with a view toward algebraic geometry.* Springer-Verlag, New York, 1995.
18. D. Eisenbud, G. Fløystad, and F.-O. Schreyer: Sheaf cohomology and free resolutions over exterior algebras. Work in progress, 2001.
19. D. Eisenbud and F.-O. Schreyer: Resultants, chow forms, and free resolutions. In preparation, 2001.
20. P. Ellia and G. Sacchiero: Smooth surfaces of $\mathbf{P}^4$ ruled in conics. In P. Newstead, editor: , *Algebraic geometry. Papers presented for the EUROPROJ conferences held in Catania, Italy, September 1993, and Barcelona, Spain, September 1994*, pages 49–62. Marcel Dekker, New York, 1994.
21. W. Fulton: *Intersection theory.* Springer-Verlag, New York, 1984.
22. D. Gieseker: On the moduli of vector bundles on an algebraic surface. *Ann. of Math. (2)*, 106:45–60, 1977.
23. G. Horrocks: Vector bundles on the punctured spectrum of a local ring. *Proc. London Math. Soc. (3)*, 14:689–713, 1964.
24. G. Horrocks: Construction of bundles on $\mathbf{P}^n$. In A. Douady and J.-L. Verdier, editors: , *Les équations de Yang-Mills*, pages 197–203. Société Mathématique de France, Paris, 1980. Séminaire E. N. S., 1977-1978, Astérisqe 71–72.
25. G. Horrocks and D. Mumford: A rank 2 vector bundle on $\mathbf{P}^4$ with 15,000 symmetries. *Topology*, 12:63–81, 1973.
26. Klaus Hulek: Stable rank-2 vector bundles on $\mathbf{P}_2$ with c_1 odd. *Math. Ann.*, 242(3):241–266, 1979.
27. P. Ionescu: Embedded projective varieties of small invariants III. In A.J. Sommese, A. Biancofiore, and E. Livorni, editors: , *Algebraic Geometry (L'Aquila 1988)*, pages 138–154. Springer, New York, 1990.
28. M. M. Kapranov: On the derived categories of coherent sheaves on some homogeneous spaces. *Invent. Math.*, 92(3):479–508, 1988.
29. S. Kleiman: Geometry on grassmanians and applications to splitting bundles and smoothing cycles. *Publ. Math. I.H.E.S.*, 36:281–297, 1969.
30. A. Lanteri: On the existence of scrolls in $\mathbf{P}^4$. *Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat.*, 69:223–227, 1980.
31. J. Le Potier: Fibrés stables de rang 2 sur $\mathbf{P}_2(\mathbf{C})$. *Math. Ann.*, 241:217–256, 1979.
32. N. Manolache: Syzygies of abelian surfaces embedded in $\mathbf{P}^4(\mathbf{C})$. *J. reine angew. Math.*, 384:180–191, 1988.

33. M. Maruyama: Moduli of stable sheaves I. *J. Math. Kyoto Univ.*, 17:91–126, 1977.
34. M. Maruyama: Moduli of stable sheaves II. *J. Math. Kyoto Univ.*, 18:557–614, 1978.
35. Ch. Okonek: Reflexive Garben auf $\mathbf{P}^4$. *Math. Ann.*, 260:211–237, 1982.
36. Ch. Okonek: Flächen vom Grad 8 im $\mathbf{P}^4$. *Math. Z.*, 191:207–223, 1986.
37. Ch. Okonek, M. Schneider, and H. Spindler: *Vector bundles on complex projective spaces*. Birkhäuser Boston, Mass., 1980.
38. L. Roth: On the projective classification of surfaces. *Proc. of London Math. Soc.*, 42:142–170, 1937.
39. J.-P. Serre: Faisceaux algébriques cohérents. *Ann. of Math. (2)*, 61:197–278, 1955.
40. Gregory G. Smith: Computing global extension modules. *J. Symbolic Comput.*, 29(4-5):729–746, 2000. Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998).
41. Wolmer V. Vasconcelos: *Computational methods in commutative algebra and algebraic geometry*. Springer-Verlag, Berlin, 1998. With chapters by David Eisenbud, Daniel R. Grayson, Jürgen Herzog and Michael Stillman.
42. C. H. Walter: Pfaffian subschemes. *J. Algebraic Geom.*, 5:671–704, 1996.

Index